Analysis of *RLC* Elements under Stochastic Conditions Using the First and the Second Moments

Janusz WALCZAK, Seweryn MAZURKIEWICZ, Dariusz GRABOWSKI
Silesian University of Technology, Faculty of Electrical Engineering
ul. Akademicka 10, 44-100 Gliwice, Poland
dariusz.grabowski@polsl.pl

***Abstract*—This paper describes a method of determining the first two moments of the response for basic components of electrical circuits, i.e. resistors, inductors and capacitors. The paper goal was to obtain closed form formulae for the moments describing voltage or current stochastic processes. It has been assumed that the element parameters *R* (resistance), *L* (inductance) and *C* (capacitance) could be random variables, deterministic functions or stochastic processes and excitations are second order stochastic processes. Moreover, two cases of dependence between the random parameters and the excitation stochastic processes have been considered. The obtained results enable determination of exact solutions for the first two moments without application of numerical algorithms.**

***Index Terms*—circuit analysis, linear circuits, moment methods, stochastic processes, stochastic systems.**

I. Introduction

Many works, including monograph [1], have been devoted to analysis of stochastic phenomena in electrical and electronic circuits. Determination of probabilistic characteristics for stochastic processes observed in systems is very often of great importance. Works in the field of stochastic system analysis can be divided into two main topic groups. The first one concerns deterministic systems in which some stochastic signal sources are present [2-4]. The second one deals with systems in which sources as well as basic elements require probabilistic description [5-11]. The analysis of such systems is usually carried out by means of stochastic differential or integral equations. However, their models can be also built with the aid of stochastic moments [12].

Analytic solutions of stochastic differential equations describing *RC*, *RL* and *RLC* electrical circuits, in which a noise term has been added to selected parameters and/or to an input signal, have been presented in [2], [4-5], [7-8] and [13]. In the case of more complex systems results have been given rather in terms of moments, especially the second-order statistics (SOS), than in a closed form. Such analysis has been performed for cascade connections of linear two-ports with randomly varied parameters [10-11]. The second-order statistics are above all effective in the case of probability density functions completely described by their first two moments, e.g. Gaussian distributions, and their transformations in linear systems. In the case of nonlinear systems with uncertain circuit elements, the statistical simulation using the polynomial chaos expansion could be applied [14-15].

This paper deals with determination of the first and the second moments for the voltage stochastic process observed in the case of random elements *R*, *L* and *C* assuming that the current stochastic process moments are given. Moreover, it has been assumed that lumped parameters *R*, *L* and *C* could be described by random variables or by functions $R(t)$, $L(t)$ and $C(t)$, which are deterministic functions or stochastic processes. This paper is a continuation of previous works devoted to determination of stochastic process moments for deterministic linear elements supplied by stochastic current sources [3], [9] and for nonlinear inertialess elements described by random polynomials [16].

Despite its theoretical character, the paper has also some practical implications. Signal analysis, system identification and signal estimation problems are very often solved using directly or indirectly stochastic moments, especially the second-order statistics. Although, the paper concentrates on analysis of basic dynamical elements, nevertheless these elements could be used to build more complex dynamical models and enable determination of stochastic moments of output signals in electric and electronic systems applicable in practice. For example, the developed formulae could be applied for fast estimation of statistical parameters of signals at outputs of signal processing units performing such operations as sampling, modulation, signal detection or filtering. Such approach has been already applied for analysis of output signals in transmission line models consisting of cascade *RLGC* branches [11], mobile-to-mobile communication channels [17], sampling mixers working at radio frequency or intermediate frequency [18], CMOS inverters at the low power supply voltage [19].

Results presented in this paper could also be applied in stochastic dynamic analysis of mechanical, geophysical and hydrological system models [20].

II. Resistor Stochastic Models

If resistance is described by a random variable *R*, then stochastic current and voltage processes for a resistor are related by:

$$U(t) = RI(t), \tag{1}$$

where *R* - random variable with given distribution.

The following two cases can be considered:

- the current process $I(t)$ and the random variable *R* are statistically independent,
- the current process $I(t)$ and the random variable *R* are not statistically independent.

Digital Object Identifier 10.4316/AECE.2015.04010

In the first case, assuming that the moments of the resistor current are known and applying expected value operator to (1) result in closed form formulae expressing the first and the second moments of the voltage stochastic process across the resistor:

$$m_U(t) = \mathrm{E}[R]\mathrm{E}[I(t)] = m_R m_I(t), \quad (2)$$

$$\sigma_U^2(t) = \mathrm{E}\left[R^2\right] \mathcal{R}_I(t,t) - m_U^2(t), \quad (3)$$

$$\mathcal{R}_U(t_1,t_2) = \mathrm{E}[U(t_1)U(t_2)] = \mathrm{E}\left[R^2\right] \mathcal{R}_I(t_1,t_2), \quad (4)$$

where:

$m_U(t) = \mathrm{E}[U(t)]$ – expected value of the process $U(t)$,

$m_I(t) = \mathrm{E}[I(t)]$ – expected value of the process $I(t)$,

$m_R(t) = \mathrm{E}[R]$ – expected value of the random variable R,

$\mathrm{E}\left[R^2\right]$ – second raw moment of the random variable R,

$\sigma_U^2 = \mathrm{E}\left[U^2(t)\right] - m_U^2(t)$ – variance of the process $U(t)$,

$\mathcal{R}_U(t_1,t_2) = \mathrm{E}[U(t_1)U(t_2)]$ – autocorrelation function of the voltage process $U(t)$,

$\mathcal{R}_I(t_1,t_2) = \mathrm{E}[I(t_1)I(t_2)]$ – autocorrelation function of the current process $I(t)$.

The analysis of (2), (3) and (4) leads to a conclusion that if the given current process $I(t)$ and the random variable R representing the resistance are independent, then the description of the resistor can be made with the aid of the first and the second moments of the current process as well as the random variable R.

In the second case, equation (2) is not valid and the following relation should be used:

$$m_U(t) = \mathrm{E}[U(t)] = \mathrm{E}[RI(t)]. \quad (5)$$

There are few methods to expand formula (5) [21]. The simplest one consists in application of the expected value definition:

$$\mathrm{E}[g(X_1(t_1),\ldots,X_n(t_n))] = \\ = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,\ldots,x_n) f_{X_1,\ldots,X_n}(x_1,\ldots,x_n,t_1,\ldots,t_n)\mathrm{d}x_1,\ldots,\mathrm{d}x_n, \quad (6)$$

where:

$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n,t_1,\ldots,t_n)$ – joint probability density function of the random variables $X_1(t_1)$, $X_2(t_2)$, …, $X_n(t_n)$ defined by the stochastic processes $X(t)$ for moments $t_1,t_2,\ldots,t_n$, respectively,

$g(x_1,\ldots x_n)$ – deterministic function of n variables.

Equation (6) can be applied to express moments of the voltage stochastic process across the resistor in the case of statistical dependence between the resistance R and the current stochastic process:

$$m_U(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ri f_{RI}(r,i,t)\mathrm{d}r\mathrm{d}i, \quad (7)$$

$$\sigma_U^2(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r^2 i^2 f_{RI}(r,i,t)\mathrm{d}r\mathrm{d}i - m_U^2(t), \quad (8)$$

$$R_U(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r^2 i_1 i_2 f_{RII}(r,i_1,i_2,t_1,t_2)\mathrm{d}r\mathrm{d}i_1\mathrm{d}i_2, \quad (9)$$

where:

$f_{RI}(r,i,t)$ – joint probability density function of the random variable R and the stochastic processes $I(t)$ for a moment t,

$f_{RII}(r,i_1,i_2,t_1,t_2)$ – joint probability density function of the random variable R and the stochastic processes $I(t)$ for moments t_1 and t_2.

In this case, moments of the input current process $I(t)$ and the random variable R describing the element parameter are not sufficient to express moments of the output voltage process. Joint probability density functions must be given.

Resistor could be also a time-varying element described by the following equation:

$$U(t) = R(t)I(t). \quad (10)$$

Three special cases could be considered, i.e. the function $R(t)$ may be:

– a deterministic function,
– a stochastic process which is statistically independent of the current processes $I(t)$,
– a stochastic process which is not statistically independent of the current processes $I(t)$.

In the first case, the first and the second moments of the output voltage process are expressed by:

$$m_U(t) = R(t)m_I(t), \quad (11)$$

$$\sigma_U^2(t) = R^2(t)\mathcal{R}_I(t,t) - m_U^2(t), \quad (12)$$

$$\mathcal{R}_U(t_1,t_2) = R(t_1)R(t_2)\mathcal{R}_I(t_1,t_2). \quad (13)$$

In the second case, the first and the second moments of the output voltage process are expressed by:

$$m_U(t) = m_R(t)\cdot m_I(t), \quad (14)$$

$$\sigma_U^2(t) = \mathrm{E}\left[R^2(t)\right] \mathcal{R}_I(t,t) - m_U^2(t), \quad (15)$$

$$\mathcal{R}_U(t_1,t_2) = \mathcal{R}_R(t_1,t_2)\mathcal{R}_I(t_1,t_2). \quad (16)$$

In the third case, the equations expressing the mean function and the variance are identical with (7) and (8). However, the interpretation of the function f_{RI} (r, i, t) is different. The function f_{RI} (r, i, t) is a joint probability density function of the processes $R(t)$ and $I(t)$ for a moment t. The relation for the autocorrelation function takes the form:

$$\mathcal{R}_U(t_1,t_2) = \\ = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} r_1 r_2 i_1 i_2 f_{RRII}(r_1,r_2,i_1,i_2,t_1,t_2)\mathrm{d}r_1\mathrm{d}r_2\mathrm{d}i_1\mathrm{d}i_2, \quad (17)$$

where:

$f_{RRII}(r_1,r_2,i_1,i_2,t_1,t_2)$ – joint probability density function of the random variable $R(t)$ and the stochastic processes $I(t)$ for moments t_1 and t_2.

III. EXAMPLE I

The case for which the random variable R and the input process $I(t)$ are independent has been considered in the example. If:

$$I(t) = A\sin(t) + W(t), \quad (18)$$

where:

A – random variable with given moments: $m_A = \mathrm{E}[A]$ and $\sigma_A^2 = \mathrm{E}\left[A^2\right] - m_A^2$,

$W(t)$ – Wiener process,

then the voltage across the resistor described by the resistance R is given by:

$$U(t) = RA\sin(t) + RW(t)\,. \tag{19}$$

Expected values of the input and the output processes are expressed by:

$$m_I(t) = m_A \sin(t)\,, \tag{20}$$

$$m_U(t) = m_R m_A \sin(t)\,. \tag{21}$$

The variances of the input and the output processes are given by:

$$\sigma_I^2(t) = \sigma_A^2(t)\sin^2(t) + t\,, \tag{22}$$

$$\sigma_U^2(t) = \sigma_{RA}^2 \sin^2(t) + \mathrm{E}\left[R^2\right]\, t\,, \tag{23}$$

where σ^2_{RA} stands for the variance of the product of independent random variables R and A:

$$\sigma_{RA}^2 = \sigma_R^2\sigma_A^2 + \sigma_R^2 m_A^2 + \sigma_A^2 m_R^2\,. \tag{24}$$

The autocorrelation functions of the input and the output processes as well as the cross correlation function of these processes can be written as:

$$\mathcal{R}_I(t_1,t_2) = \mathrm{E}\left[A^2\right]\ \sin(t_1)\sin(t_2) + \min(t_1,t_2)\,, \tag{25}$$

$$\mathcal{R}_{IU}(t_1,t_2) = m_R \mathrm{E}\left[A^2\right]\ \sin(t_1)\sin(t_2) + m_R\ \min(t_1,t_2)\,, \tag{26}$$

$$\mathcal{R}_U(t_1,t_2) = \mathrm{E}\left[R^2\right]\mathrm{E}\left[A^2\right]\ \sin(t_1)\sin(t_2) + \mathrm{E}\left[R^2\right]\min(t_1,t_2)\,. \tag{27}$$

IV. INDUCTOR STOCHASTIC MODELS

Illustrations and tables should be progressively numbered, following the order cited in the text; they may be organized

Stochastic current and voltage processes in the case of an inductor whose the inductance is a random variable are related by:

$$U(t) = L\frac{\mathrm{d}I(t)}{\mathrm{d}t}\,, \tag{28}$$

where:

L – random variable with given distribution,

$U(t)$ – voltage stochastic process of the inductor,

$I(t)$ – current stochastic process of the inductor.

The following two cases can be considered (similar to the cases analyzed for the resistor):

– the current process and the random variable L are statistically independent,

– the current process and the random variable L are not statistically independent.

In the first case, assuming that the moments of the inductor current are known and applying expected value operator to (28) results in closed form formulae expressing the first and the second moments of the voltage across the inductor:

$$m_U(t) = m_L \frac{\mathrm{d}m_I(t)}{\mathrm{d}t}\,, \tag{29}$$

$$\sigma_U^2(t) = \mathrm{E}\left[L^2\right]\cdot \frac{\partial^2}{\partial t_1\,\partial t_1}\mathcal{R}_1(t_1,t_2)\bigg|_{t_1=t_2=t} - m_U^2(t)\,, \tag{30}$$

$$\mathcal{R}_U(t_1,t_2) = \mathrm{E}\left[L^2\right]\cdot \frac{\partial^2}{\partial t_1\,\partial t_1}\mathcal{R}_1(t_1,t_2)\,. \tag{31}$$

In the second case, the definition of the expected value operator (6) must be applied. The first and the second moments can be expressed by the equations:

$$m_U(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} l\, i\, f_{LI}(l,i,t)\mathrm{d}l\,\mathrm{d}i\,, \tag{32}$$

$$\sigma_U^2(t) = \\ = \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} l^2 i_1 i_2 f_{LII}(l,i_1,i_2,t_1,t_2)\mathrm{d}l\,\mathrm{d}i_1\mathrm{d}i_2\bigg|_{t_1=t_2=t} \\ - m_U^2(t)\,, \tag{33}$$

$$\mathcal{R}_U(t_1,t_2) = \\ = \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} l^2 i_1 i_2 f_{LII}(l,i_1,i_2,t_1,t_2)\mathrm{d}l\,\mathrm{d}i_1\mathrm{d}i_2\,. \tag{34}$$

The model of a time-varying inductor is described by the following equation:

$$U(t) = L(t)\frac{\mathrm{d}I(t)}{\mathrm{d}t} + I(t)\frac{\mathrm{d}L(t)}{\mathrm{d}t}\,. \tag{35}$$

Three special cases have been considered - the function $L(t)$ has been assumed to be:

– a deterministic function,

– a stochastic process which is statistically independent of the current processes $I(t)$,

– a stochastic process which is not statistically independent of the current processes $I(t)$.

In the first case, the first and the second moments of the output voltage process are expressed by:

$$m_U(t) = L(t)\frac{\mathrm{d}m_I(t)}{\mathrm{d}t} + m_I(t)\frac{\mathrm{d}L(t)}{\mathrm{d}t}\,, \tag{36}$$

$$\sigma_U^2(t) = L^2(t)\sigma_{I'}^2(t) + \\ + 2L(t)\frac{\mathrm{d}L(t)}{\mathrm{d}t}\left(\frac{\partial}{\partial t_1}\mathcal{R}_I(t_1,t_2)\bigg|_{t_1=t_2=t} - m_I(t)\frac{\mathrm{d}m_I(t)}{\mathrm{d}t}\right) + \\ + \left(\frac{\mathrm{d}L(t)}{\mathrm{d}t}\right)^2\sigma_I^2(t)\,, \tag{37}$$

$$\mathcal{R}_U(t_1,t_2) = L(t_1)L(t_2)\frac{\partial^2}{\partial t_1\partial t_2}\mathcal{R}_I(t_1,t_2) + \\ + \frac{\mathrm{d}L(t)}{\mathrm{d}t}\bigg|_{t=t_1} L(t_2)\frac{\partial}{\partial t_2}\mathcal{R}_I(t_1,t_2) + \\ + L(t_1)\frac{\mathrm{d}L(t)}{\mathrm{d}t}\bigg|_{t=t_2}\frac{\partial}{\partial t_1}\mathcal{R}_I(t_1,t_2) + \\ + \frac{\mathrm{d}L(t)}{\mathrm{d}t}\bigg|_{t=t_1}\frac{\mathrm{d}L(t)}{\mathrm{d}t}\bigg|_{t=t_2}\mathcal{R}_I(t_1,t_2)\,, \tag{38}$$

where:

$$\sigma_{I'}^2(t) = \mathrm{E}\left[(I'(t))^2\right] - \left(\mathrm{E}\left[I'(t)\right]\right)^2\,.$$

In the second case, the first and the second moments of the output voltage process are expressed by:

$$m_U(t) = m_L(t)\frac{\mathrm{d}m_I(t)}{\mathrm{d}t} + m_I(t)\frac{\mathrm{d}m_L(t)}{\mathrm{d}t}\,, \tag{39}$$

$$\sigma_U^2(t) = \left(\mathcal{R}_L(t_1,t_2) \frac{\partial^2}{\partial t_1 \partial t_2} \mathcal{R}_I(t_1,t_2) \right)\Bigg|_{t_1=t_2=t} - $$
$$- m_L^2(t)\left(\frac{\mathrm{d}m_I(t)}{\mathrm{d}t}\right)^2 - m_I^2(t)\left(\frac{\mathrm{d}m_L(t)}{\mathrm{d}t}\right)^2 +$$
$$+2\left(\frac{\partial}{\partial t_1}\left(\mathcal{R}_L(t_1,t_2)\right)\frac{\partial}{\partial t_2}\left(\mathcal{R}_I(t_1,t_2)\right)\right)\Bigg|_{t_1=t_2=t} - \quad (40)$$
$$-2m_I(t)\frac{\mathrm{d}m_I(t)}{\mathrm{d}t} m_L(t)\frac{\mathrm{d}m_L(t)}{\mathrm{d}t} +$$
$$+\left(\mathcal{R}_I(t_1,t_2)\frac{\partial^2}{\partial t_1 \partial t_2}\mathcal{R}_L(t_1,t_2)\right)\Bigg|_{t_1=t_2=t},$$

$$\mathcal{R}_U(t_1,t_2) = \mathcal{R}_L(t_1,t_2)\frac{\partial^2}{\partial t_1 \partial t_2}\mathcal{R}_I(t_1,t_2) +$$
$$+\frac{\partial}{\partial t_1}\left(\mathcal{R}_L(t_1,t_2)\right)\frac{\partial}{\partial t_2}\left(\mathcal{R}_I(t_1,t_2)\right) +$$
$$+\frac{\partial}{\partial t_2}\left(\mathcal{R}_L(t_1,t_2)\right)\frac{\partial}{\partial t_1}\left(\mathcal{R}_I(t_1,t_2)\right) + \quad (41)$$
$$+\frac{\partial^2}{\partial t_1 \partial t_2}\left(\mathcal{R}_L(t_1,t_2)\right)\mathcal{R}_I(t_1,t_2).$$

In the third case, the equations expressing the mean function and the variance are identical with (32) and (33). However, the interpretation of the function f_{LI} (l, i, t) is different – it is a joint probability density function of the processes $L(t)$ and $I(t)$ for a moment t. The relation for the autocorrelation function takes the form:

$$\mathcal{R}_U(t_1,t_2) =$$
$$= \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} l_1 l_2 i_1 i_2 f_{LLII}(l_1,l_2,i_1,i_2,t_1,t_2)\mathrm{d}l_1\mathrm{d}l_2\mathrm{d}i_1\mathrm{d}i_2 \quad (42)$$

where:

$f_{LLII}(l_1,l_2,i_1,i_2,t_1,t_2)$ – joint probability density function of the random variable $L(t)$ and the stochastic processes $I(t)$ for moments t_1 and t_2.

V. Example II

Let us assume that the random variable L and the input process $I(t)$ are independent. Moreover, the input process is defined like in the previous example:

$$I(t) = A\sin(t) + W(t), \quad (43)$$

where:

A – random variable with given moments: $m_A = \mathrm{E}[A]$ and $\sigma_A^2 = \mathrm{E}\left[A^2\right] - m_A^2$,

$W(t)$ – Wiener process,

The voltage stochastic process across the inductor is equal:

$$U(t) = LA\cos(t) + LN(t), \quad (44)$$

where $N(t)$ – white noise.

Expected values of the input and the output processes are expressed by:

$$m_I(t) = m_A\sin(t), \quad (45)$$

$$m_U(t) = m_L m_A\cos(t). \quad (46)$$

The variances of the input and the output processes are given by:

$$\sigma_I^2(t) = \sigma_A^2(t)\sin^2(t) + t, \quad (47)$$

$$\sigma_U^2(t) = \infty. \quad (48)$$

The autocorrelation functions of the input and the output processes can be written as:

$$\mathcal{R}_I(t_1,t_2) = \mathrm{E}\left[A^2\right]\sin(t_1)\sin(t_2) + \min(t_1,t_2), \quad (49)$$

$$\mathcal{R}_U(t_1,t_2) = \mathrm{E}\left[L^2\right]\mathrm{E}\left[A^2\right]\cos(t_1)\cos(t_2) + \mathrm{E}\left[L^2\right]\delta(t_1 - t_2). \quad (50)$$

VI. Capacitor Stochastic Models

Stochastic current and voltage processes in the case of a capacitor, whose capacitance is a random variable, are related by:

$$I(t) = C\frac{\mathrm{d}U(t)}{\mathrm{d}t}, \quad (51)$$

where:

C – random variable with given distribution,

$U(t)$ – voltage stochastic process of the capacitor,

$I(t)$ – current stochastic process of the capacitor.

The following two cases can be considered (similar to the cases analyzed for the resistor):

- the voltage process and the random variable C are statistically independent,
- the voltage process and the random variable C are not statistically independent.

In the first case, assuming that the capacitor voltage is known and applying expected value operator to (51) results in closed form formulae expressing the first and the second moments of the voltage across the capacitor:

$$m_I(t) = m_C\frac{\mathrm{d}m_U(t)}{\mathrm{d}t}, \quad (52)$$

$$\sigma_I^2(t) = \mathrm{E}\left[C^2\right]\frac{\partial^2}{\partial t_1 \partial t_2}\mathcal{R}_U(t_1,t_2)\Bigg|_{t_1=t_2=t} - m_I^2(t), \quad (53)$$

$$\mathcal{R}_I(t_1,t_2) = \mathrm{E}\left[C^2\right]\frac{\partial^2}{\partial t_1 \partial t_2}\mathcal{R}_U(t_1,t_2). \quad (54)$$

In the second case, the definition of the expected value operator (6) must be applied. The first and the second moments can be expressed by the equations:

$$m_I(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} cu f_{CU}(c,u,t)\mathrm{d}c\mathrm{d}u, \quad (55)$$

$$\sigma_I^2(t) =$$
$$= \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c^2 u_1 u_2 f_{CUU}(c,u_1,u_2,t_1,t_2)\mathrm{d}c\mathrm{d}u_1\mathrm{d}u_2\Bigg|_{t_1=t_2=t} -$$
$$- m_I^2(t), \quad (56)$$

$$\mathcal{R}_I(t_1,t_2) =$$
$$= \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c^2 u_1 u_2 f_{CUU}(c,u_1,u_2,t_1,t_2)\mathrm{d}c\mathrm{d}u_1\mathrm{d}u_2. \quad (57)$$

The model of a time-varying capacitor can be described by the following equation:

$$I(t) = C(t)\frac{\mathrm{d}U(t)}{\mathrm{d}t} + U(t)\frac{\mathrm{d}C(t)}{\mathrm{d}t}. \tag{58}$$

Three special cases have been considered - the function $C(t)$ has been assumed to be:

– a deterministic function,
– a stochastic process which is statistically independent of the voltage processes $U(t)$,
– a stochastic process which is not statistically independent of the voltage processes $U(t)$.

In the first case, the first and the second moments of the output current process are expressed by:

$$m_I(t) = C(t)\frac{\mathrm{d}m_U(t)}{\mathrm{d}t} + m_U(t)\frac{\mathrm{d}C(t)}{\mathrm{d}t}, \tag{59}$$

$$\begin{aligned}\sigma_I^2(t) &= C^2(t)\sigma_{U'}^2(t) + \\ &+ 2C(t)\frac{\mathrm{d}C(t)}{\mathrm{d}t}\left(\frac{\partial}{\partial t_1}\mathscr{R}_U(t_1,t_2)\bigg|_{t_1=t_2=t} - m_U(t)\frac{\mathrm{d}m_U(t)}{\mathrm{d}t}\right) + \\ &+ \left(\frac{\mathrm{d}C(t)}{\mathrm{d}t}\right)^2\sigma_U^2(t),\end{aligned} \tag{60}$$

$$\begin{aligned}\mathscr{R}_I(t_1,t_2) &= C(t_1)\,C(t_2)\frac{\partial^2}{\partial t_1\partial t_2}\mathscr{R}_U(t_1,t_2) + \\ &+ \frac{\mathrm{d}C(t)}{\mathrm{d}t}\bigg|_{t=t_1} C(t_2)\frac{\partial}{\partial t_2}\mathscr{R}_U(t_1,t_2) + \\ &+ C(t_1)\frac{\mathrm{d}C(t)}{\mathrm{d}t}\bigg|_{t=t_2}\frac{\partial}{\partial t_1}\mathscr{R}_U(t_1,t_2) + \\ &+ \frac{\mathrm{d}C(t)}{\mathrm{d}t}\bigg|_{t=t_1}\frac{\mathrm{d}C(t)}{\mathrm{d}t}\bigg|_{t=t_2}\mathscr{R}_U(t_1,t_2),\end{aligned} \tag{61}$$

where:

$\sigma_{U'}^2(t) = \mathrm{E}\left[(U'(t))^2\right] - \left(\mathrm{E}[U'(t)]\right)^2$.

In the second case, the first and the second moments of the output current process are expressed by:

$$m_I(t) = m_C(t)\frac{\mathrm{d}m_U(t)}{\mathrm{d}t} + m_U(t)\frac{\mathrm{d}m_C(t)}{\mathrm{d}t}, \tag{62}$$

$$\begin{aligned}\sigma_I^2(t) &= \left(\mathscr{R}_C(t_1,t_2)\frac{\partial^2}{\partial t_1\partial t_2}\mathscr{R}_C(t_1,t_2)\right)\bigg|_{t_1=t_2=t} - \\ &- m_C^2(t)\left(\frac{\mathrm{d}m_U(t)}{\mathrm{d}t}\right)^2 + \\ &+ 2\left(\frac{\partial}{\partial t_1}\left(\mathscr{R}_C(t_1,t_2)\right)\frac{\partial}{\partial t_2}\left(\mathscr{R}_U(t_1,t_2)\right)\right)\bigg|_{t_1=t_2=t} - \\ &- 2m_U(t)\frac{\mathrm{d}m_U(t)}{\mathrm{d}t}m_C(t)\frac{\mathrm{d}m_C(t)}{\mathrm{d}t} + \\ &+ \left(\mathscr{R}_U(t_1,t_2)\frac{\partial^2}{\partial t_1\partial t_2}\mathscr{R}_C(t_1,t_2)\right)\bigg|_{t_1=t_2=t} - \\ &- m_U^2(t)\left(\frac{\mathrm{d}m_C(t)}{\mathrm{d}t}\right)^2,\end{aligned} \tag{63}$$

$$\begin{aligned}\mathscr{R}_I(t_1,t_2) &= \mathscr{R}_C(t_1,t_2)\frac{\partial^2}{\partial t_1\partial t_2}\mathscr{R}_U(t_1,t_2) + \\ &+ \frac{\partial}{\partial t_1}\left(\mathscr{R}_C(t_1,t_2)\right)\frac{\partial}{\partial t_2}\left(\mathscr{R}_U(t_1,t_2)\right) + \\ &+ \frac{\partial}{\partial t_2}\left(\mathscr{R}_C(t_1,t_2)\right)\frac{\partial}{\partial t_1}\left(\mathscr{R}_U(t_1,t_2)\right) + \\ &+ \frac{\partial^2}{\partial t_1\partial t_2}\left(\mathscr{R}_C(t_1,t_2)\right)\mathscr{R}_U(t_1,t_2).\end{aligned} \tag{64}$$

In the third case, the equations expressing the mean function and the variance are identical with (55) and (56). However, the interpretation of the function f_{CU} (c, u, t) is different. The function f_{CU} (c, u, t) is a joint probability density function of the processes $C(t)$ and $U(t)$ for a moment t. The relation for the autocorrelation function takes the form:

$$\mathscr{R}_I(t_1,t_2) = \frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c_1c_2u_1u_2 f_{CCUU}(c_1,c_2,u_1,u_2,t_1,t_2)\mathrm{d}c_1\mathrm{d}c_2\mathrm{d}u_1\mathrm{d}u_2 \tag{65}$$

where:

$f_{CCUU}(c_1,c_2,u_1,u_2,t_1,t_2)$ – joint probability density function of the random variable $C(t)$ and the stochastic processes $U(t)$ for moments t_1 and t_2.

VII. Example III

Let us assume that the random variable C and the input process $U(t)$ are independent. Moreover, the input process is defined as:

$$U(t) = \sin(\Omega t) + W(t), \tag{66}$$

where:

Ω – random variable with uniform distribution:

$$f_\Omega(x) = \frac{1}{b_\Omega - a_\Omega}\left(H(x-a_\Omega) - H(x-b_\Omega)\right), \tag{67}$$

where $H(x)$ – Heaviside step function.

The current stochastic process for the capacitor is equal:

$$I(t) = C\Omega\cos(\Omega t) + CN(t). \tag{68}$$

Expected values of the input and the output processes are expressed by:

$$m_U(t) = \frac{1}{b_\Omega - a_\Omega}\frac{\cos(a_\Omega t) - \cos(b_\Omega t)}{t}, \tag{69}$$

$$\begin{aligned}m_I(t) &= m_C\frac{1}{b_\Omega - a_\Omega}\cdot \\ &\cdot\left(\frac{-a_\Omega\sin(a_\Omega t) + b_\Omega\sin(b_\Omega t)}{t} - \frac{\cos(a_\Omega t) - \cos(b_\Omega t)}{t}\right).\end{aligned} \tag{70}$$

It can be proved using l'Hopital's rule that $m_U(0) = 0$ and $|m_I(0)| < \infty$.

The autocorrelation functions of the input and the output processes for $t_1 \neq t_2$ can be written as:

$$\begin{aligned}\mathscr{R}_U(t_1,t_2) &= \frac{1}{2\left(t_1^2 - t_2^2\right)}\big((-t_1-t_2)\sin(a_\Omega(t_1-t_2)) + \\ &+ (t_1+t_2)\sin(b_\Omega(t_1-t_2)) + (t_1-t_2)\sin(a_\Omega(t_1-t_2)) \\ &+ (-t_1-t_2)\sin(b_\Omega(t_1-t_2))\big) + \min(t_1,t_2),\end{aligned} \tag{71}$$

$$\underset{t_1 \neq t_2}{\forall} \mathscr{R}_I(t_1,t_2) = \frac{1}{2\left(t_1 - t_2\right)^3 \left(t_1 + t_2\right)^3} \cdot$$
$$\cdot(-(t_1+t_2)^3(a_\Omega^2 t_1^2 - 2a_\Omega^2 t_1 t_2 + a_\Omega^2 t_2^2 - 2)\sin(a_\Omega(t_1-t_2))$$
$$+(t_1+t_2)^3(b_\Omega^2 t_1^2 - 2b_\Omega^2 t_1 t_2 + b_\Omega^2 t_2^2 - 2)\sin(b_\Omega(t_1-t_2))$$
$$-(t_1-t_2)(2a_\Omega(t_1+t_2)^3\cos(a_\Omega(t_1-t_2))$$
$$-2b_\Omega(t_1+t_2)^3\cos(b_\Omega(t_1-t_2))$$
$$+(t_1-t_2)^2((a_\Omega^2 t_1^2 + 2a_\Omega^2 t_1 t_2 + a_\Omega^2 t_2^2 - 2)\sin(a_\Omega(t_1+t_2))$$
$$-(b_\Omega^2 t_1^2 + 2b_\Omega^2 t_1 t_2 + b_\Omega^2 t_2^2 - 2)\sin(b_\Omega(t_1+t_2))$$
$$+2(t_1+t_2)(-b_\Omega\cos(b_\Omega(t_1+t_2)) + a_\Omega\cos(a_\Omega(t_1+t_2)))))). \tag{72}$$

VIII. Complementary Equations

As in the previous sections, equations expressing the first and the second moments can be also found after the current and the voltage process role reversal (the input quantity becomes the output one and vice versa), i.e. for the complementary equations of basic electrical elements.

For example, in the case of an inductor described by:

$$I(t) = \frac{1}{L}\int_{t_0}^{t} U(t)\mathrm{d}t + I(t_0), \quad f_L(0) = 0, \tag{74}$$

where $f_L(l)$ – probability density function of the variable L, the determination of the response process moments can be based on probability density functions. If the random variable L is independent of the input process $U(t)$, then assuming zero initial conditions the moments of the current response process $I(t)$ are given by:

$$m_I(t) = \int_{-\infty}^{\infty} \frac{1}{l} f_L(l)\mathrm{d}l \int_{t_0}^{t} m_U(t)\mathrm{d}t, \tag{75}$$

$$\sigma_I^2(t) = \int_{-\infty}^{\infty} \frac{1}{l^2} f_L(l)\mathrm{d}l \int_{t_0}^{t}\int_{t_0}^{t} \mathscr{R}_U(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2 - m_I^2(t), \tag{76}$$

$$\mathscr{R}_I(t_1,t_2) = \int_{-\infty}^{\infty} \frac{1}{l^2} f_L(l)\mathrm{d}l \int_{t_0}^{t_2}\int_{t_0}^{t_1} \mathscr{R}_U(t_1,t_2)\mathrm{d}t_1\mathrm{d}t_2. \tag{77}$$

IX. Conclusion

Methods which enable calculation of expected values, variances and correlation functions for processes observed in the case of random elements R, L and C as well as $R(t)$, $L(t)$ and $C(t)$ have been described in the paper. If the random variable or stochastic process describing the element parameter and the input stochastic process are independent, then the output stochastic process moments can be determined only on the base of the moments of the input process and the moments of the parameter random variable (for stationary elements) or the moments of the parameter stochastic process (for time-varying elements). Otherwise, the joint probability density functions must be used.

References

[1] K. Skowronek, Stochastic Approach to Electrical Circuits. Monograph, Poznan: Publishing House of Poznan University of Technology, pp. 34-76, 2011.

[2] E. Kadlecova, R. Kubasek, E. Kolarova, "RL circuits modeling with noisy parameters," in Proc. of the Int. Conf. on Applied Electronics, Pilsen, 2006, pp. 79-81. [Online]. Available: http://dx.doi.org/10.1109/AE.2006.4382969

[3] D. Grabowski, "Moments of stochastic power processes for basic linear elements," in Proc. of the Int. Conf. on Fundamentals of Electrotechnics and Circuit Theory, Ustron, 2009, pp. 83-84.

[4] N. Patil, B. Gawalwad, S. Sharma, "A random input-driven resistor-capacitor series circuit," in Proc. of the Int. Conf. on Recent Advancements in Electrical, Electronics and Control Engineering, Sivakasi, 2011, pp. 100–103. [Online]. Available: http://dx.doi.org/10.1109/iconraeece.2011.6129733

[5] E. Kolarova, "Modeling RL electrical circuits by stochastic differential equations," in Proc. of the Int. Conf. Computer as a Tool, Belgrade, 2005, pp. 1236-1238. [Online]. Available: http://dx.doi.org/10.1109/EURCON.2005.1630179

[6] R. Banchuin, R. Chaisricharoen, "Stochastic inductance model of on chip active inductor," in Proc. of the Int. Conf. Education Technology and Computer, vol. 5, Shanghai, 2010, pp. V5-1 - V5-5. [Online]. Available: http://dx.doi.org/10.1109/icetc.2010.5529957

[7] R. Farnoosh, P. Nabati, R. Rezaeyan, M. Ebrahimi, "A stochastic perspective of RL electrical circuit using different noise terms," COMPEL - The International Journal for Computation and Mathematics in Electrical and Electronic Engineering, vol. 30, no. 2, pp. 812-822, 2011. [Online]. Available: http://dx.doi.org/10.1108/03321641111101221

[8] E. Kolarova, L. Brancik, "Vector linear stochastic differential equations and their applications to electrical networks," in Proc. of the 35th Int. Conf. on Telecommunications and Signal Processing, Prague, 2012, pp. 311-315. [Online]. Available: http://dx.doi.org/10.1109/tsp.2012.6256305

[9] J. Walczak, S. Mazurkiewicz, D. Grabowski, "Stochastic models of lumped elements," in Proc. of the Int. Symp. Theoretical Electrical Engineering, Pilsen, 2013, pp. II-19 – II-20.

[10] L. Brancik, E. Kolarova, "Simulation of higher-order electrical circuits with stochastic parameters via SDEs," Advances in Electrical and Computer Engineering, vol. 13, no. 1, pp. 17-22, 2013. [Online]. Available: http://dx.doi.org/10.4316/AECE.2013.01003

[11] L. Brancik, E. Kolarova, "Time-domain simulation of transmission line models with multiple stochastic excitations," in Proc. of the 24th Int. Conf. Radioelektronika, Bratislava, 2014, pp. 1–4. [Online]. Available: http://dx.doi.org/10.1109/radioelek.2014.6828407

[12] L. Socha, Linearization Methods for Stochastic Dynamic Systems. Berlin Heidelberg: Springer, Lecture Notes in Physics, pp. 59-84, 2008. [Online]. Available: http://dx.doi.org/10.1007/978-3-540-72997-6

[13] T. K. Rawat, H. Parthasarathy, "On stochastic modelling of linear circuits," International Journal of Circuit Theory and Applications, vol. 38, no. 3, pp. 259–274, 2010.

[14] P. Manfredi, D. Vande Ginste, D. De Zutter, F. G. Canavero, "Stochastic modeling of nonlinear circuits via SPICE-compatible spectral equivalents," IEEE Trans. on Circuits and Systems I: Regular Papers, vol. 61, no. 7, pp. 2057-2065, 2014. [Online]. Available: http://dx.doi.org/10.1109/TCSI.2014.2304667

[15] P. Manfredi, I. S. Stievano, F. G. Canavero, "Stochastic simulation of integrated circuits with nonlinear black-box components via augmented deterministic equivalents," Advances in Electrical and Computer Engineering, vol. 14, no. 4, pp. 3-8, 2014. [Online]. Available: http://dx.doi.org/10.4316/AECE.2014.04001

[16] D. Grabowski, "Stochastic power process for nonlinear inertialess elements," Przeglad Elektrotechniczny, vol. 86, no. 4, pp. 147-150, 2010.

[17] M. M. Olama, S. M. Djouadi, C. D. Charalambous, "Stochastic differential equations for modeling, estimation and identification of mobile-to-mobile communication channels," IEEE Trans. on Wireless Communications, vol. 8, no. 4, pp. 1754-1763, 2009. [Online]. Available: http://dx.doi.org/10.1109/TWC.2009.071068

[18] Wei Yu; B. H. Leung, "Noise analysis for sampling mixers using stochastic differential equations," IEEE Trans. on Circuits and Systems II: Analog and Digital Signal Proc., vol. 46, no. 6, pp. 699-704, 1999. [Online]. Available: http://dx.doi.org/10.1109/82.769778

[19] B. G. Gawalwad, S. N. Sharma, "Noise analysis of a CMOS inverter using the Itô stochastic differential equation," in Proc. of the IEEE Int. Conf. on Control Applications, Dubrovnik, 2012, pp. 344-349. [Online]. Available: http://dx.doi.org/10.1109/CCA.2012.6402336

[20] K. Sobczyk, Stochastic Differential Equations with Applications to Physics and Engineering. Kluwer Academic Publishers, pp. 339-364, 2001. [Online]. Available: http://dx.doi.org/10.1007/978-94-011-3712-6

[21] T. T. Soong, Random Differential Equations in Science and Engineering. New York: Academic Press, pp. 6-32, 1973.