Alleviating Border Effects in Wavelet Transforms for Nonlinear Time-varying Signal Analysis

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***Abstract*—Border effects are very common in many finite signals analysis and processing approaches using convolution operation. Alleviating the border effects that can occur in the processing of finite-length signals using wavelet transform is considered in this paper. Traditional methods for alleviating the border effects are suitable to compression or coding applications. We propose an algorithm based on Fourier series which is proved to be appropriate to the application of time-frequency analysis of nonlinear signals. Fourier series extension method preserves the time-varying characteristics of the signals. A modified signal duration expression for measuring the extent of border effects region is presented. The proposed algorithm is confirmed to be efficient to alleviate the border effects in comparison to the current methods through the numerical examples.**

***Index Terms*—Convolution, Fourier series, Frequency estimation, Spectrogram, Wavelet transforms.**

I. Introduction

The wavelet transform has been found useful for analyzing signals which are nonlinear or nonstationary due to its ability to localize a signal simultaneously in both time and frequency in a significant different way from the tradition analysis tools, such as Fourier transform or short time Fourier transform [1]–[4]. The continuous wavelet transform is defined by a convolution of the input signal with wavelet functions generated from the mother wavelet by scaling and translation. For a finite signal, convolution operation cannot be done at the ends of the signal if there is no any preprocesses of the original signal. Since, at both borders of the signal, the analysis wavelet extends into a region with no available data. In most practical applications, the signals are over a finite interval. The wavelet transform will require the computation of non-existent values outside the interval. This creates a border effect, where transform values close to the border of the signal are tainted by the unavailable data of the signal edge. Similar problems also arise in the context where a convolution is performed on finite-length signals [5]–[8].

There are many different approaches to handling borders. One alternative is to extend the signal in some suitable ways and then apply the standard wavelet transform to the extension. Several methods have been proposed to extend a signal and thereby allow for the computation of the wavelet coefficients near the edges (start and end) of the signal. The main difficulty is that distortion would appear when the extension method is not proper [9]–[16].

Traditional extension techniques include extension by zero padding, by periodicity and by symmetry. Each method has its drawbacks [12], [15], [16]. It has been shown that computing the wavelet transform of an extension signal is equivalent to using the corresponding boundary wavelets [14]–[16]. The boundary wavelets of zero padding and periodic extension have no vanishing moments at the borders. Therefore, the transform values behave as if signal was discontinuous at the borders. And boundary wavelets of symmetric extension have one vanishing moment and avoid the discontinuous at the borders. So it introduces a jump in the first derivation. However, if the reflection is symmetric the wavelets must be symmetric to ensure no distortion in the transform values. It is well known that Haar is the only symmetric wavelet with a compact support that has been found so far. One goal of this paper is to seek an extension scheme that preserves the property of vanishing moments.

Many existing extension methods are exploited to the application of data compression or coding [10], [17], [18]. In such applications, they have paid much attention on the procedures of analysis and synthesis using filter banks [9] – [11], [19]. However, for other applications, such as diseases diagnosis [1] and machine condition monitoring [2] [3], it is desired to analyze the time-frequency content of arbitrary nonlinear and nonstationary signals. And traditional methods are not appropriate for such applications. They only make simple assumption about the signal characteristics outside the borders [15], [20], [21] so that they fail to produce satisfactory results. Thus, we will show the failure of tradition methods and present an extension mode that is suited for the application of time-frequency of nonlinear signals. Our method preserves the time-varying characteristics of the signals while reduces the distortion due to improper extensions at the borders.

In some literatures [6], the extent of these border effects regions has been mentioned, but it has not been given an explicit definition. Therefore, we will show that the extent of border effects region is not equivalent to the width of wavelets under traditional mean square definition.

The paper is organized as follows. In Section II, a brief review of the border effects in the wavelets transform and the shortcomings of traditional extension methods in the application to nonlinear time-varying signal analysis are given. Section III describes the details of the Fourier extension method and illustrates the proposed approach is suitable to the application of time-frequency analysis.

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Comparisons are made with symmetric extension method. Section IV describes the modifications to traditional mean square definition the duration of wavelet that allow evaluating the extent of border effects region. Section V concludes the paper.

II. Border Effects in the Wavelet Spectrogram

The goal of time-frequency analysis has primarily been to characterize and visualize the behavior of nonstationary signals. This is achieved by abstracting both the amplitude and phase information of a signal's time series to present an image of the variation of the frequency content of the signal with respect to time. Wavelet spectrogram, which refers to a time-scale energy distribution, is such an image. It is defined as square modulus of the wavelet coefficients. Essentially, spectrogram is one of the joint time-frequency representations. The wavelet spectrogram has been widely used for biologic, medical or mechanical signal analysis, since it is particularly helpful in tackling problems involving signal identification and detection of hidden transients that is hard to detect. Wavelet spectrogram suffers from border effects which would seriously affect the consequent identification procedure. But most previously suggested methods aim at preserving the perfect reconstruction. Thus this section will discuss the effect of traditional extension on the wavelet spectrogram in order to seek a suited extension method that can alleviate border effects. These traditional methods include zero padding, periodic extension and symmetric extension.

To show the border effects of various traditional methods, we consider a nonlinear frequency-modulated signal with instantaneous frequency given by

$$f(t) = f_0 [\frac{f_1}{f_0}]^{\frac{t}{3}} \tag{1}$$

with sampling interval $T = 0.01$s and total signal length $N = 300$ in interval $t \in [0,3]$ (s). The values of parameters $f_0 = 10$, $f_1 = 40$ are chosen. This is a typical nonstationary signal. Fig. 1(a) plots the time-domain representation of the signal. The analyzing wavelet used to generate the spectrogram is a Morlet wavelet since it is very useful and common for the detection of nonlinear characteristics [22], [23]. Fig. 1(b) shows the contour of spectrogram of the test signal which exhibits marked border effects that result from extensions by zeros, which amounts to ignoring the need for extensions. Significant artifacts marked by circled can be observed near the borders of the wavelet spectrogram. In order to observe the details about the resulting spectrogram, we extract the ridge [24] of this wavelet spectrogram as shown in Fig. 1(c).It can be see that the middle part of ridge almost perfect coincides with the theoretical result, which is denoted by solid line, while the border parts are deviated from that. It should be noted that deviation of the right end is much less than that of the left. This is due to the high frequency in the right side which corresponds to short wavelet length and short border effects region that will be discussed later.

Since the three methods only have big different at the border of the ridge, it is more clear to plot one border part. In Fig. 2, we show a comparative border effects caused by the three different extension schemes. It can be observed that the zero padding might have similar performance to periodic extension, whereas the symmetric extension presents a slight better result than the other two. This graphic outcome is also confirmed by the error between theoretical result and the results provided by traditional methods shown in Fig. 2(b). All three errors decrease when it gets close to the interior of the signal where the number of adding points participating calculation also decreases.

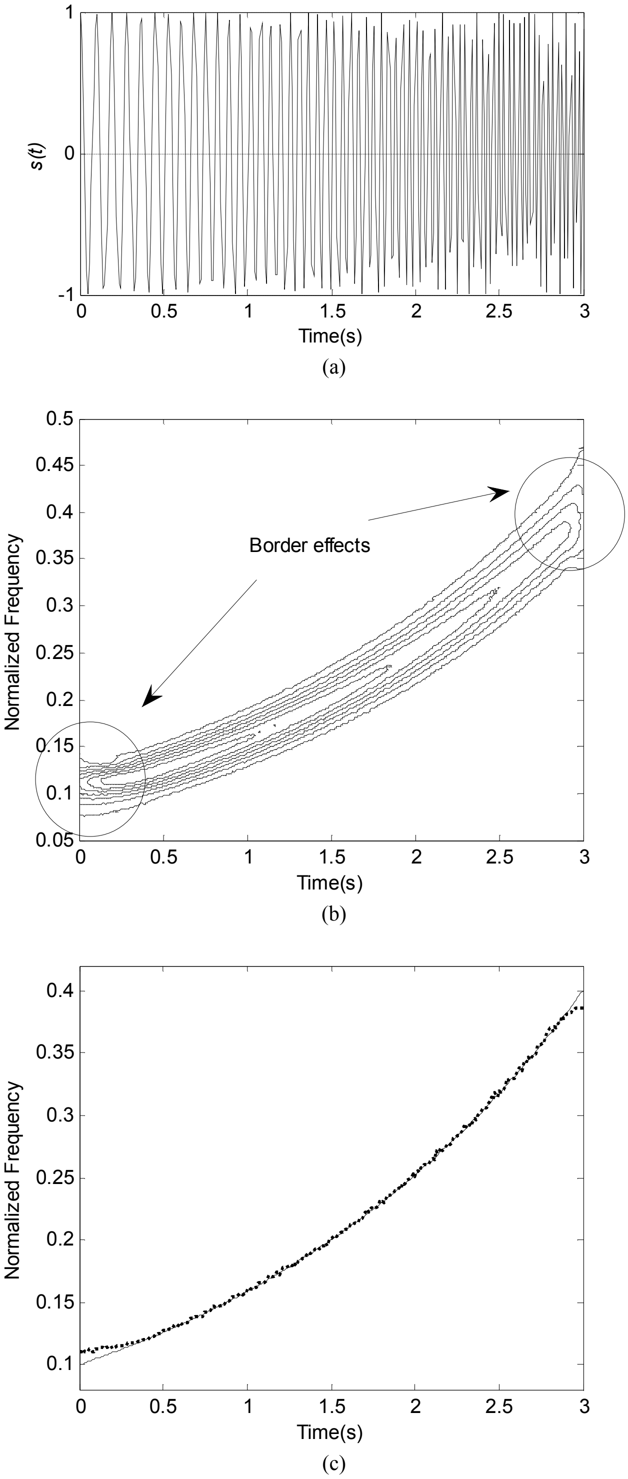


Figure 1. (a) Linear frequency modulation signal. (b) Wavelet spectrogram contour. (c) Wavelet ridge. Solid line—theoretical result; dotted line—zero padding result.

One of the main problems observed when handling borders using periodic extension is that, unless the input sequence is truly periodic and the end-points of the sequence match at the borders, artificial singularities can be introduced to wavelet transform near the borders. This is due to the discontinuity of the input sequence at the borders. Symmetric extension has the advantage, compared with periodic extension, that the extension sequence near the borders is continuous. Symmetric extension for handling borders is used extensively in many applications. It is the default extension scheme used in some software packages (e.g. MATLAB). However, the first and higher-order derivatives of the extension signal at the borders may not be continuous.

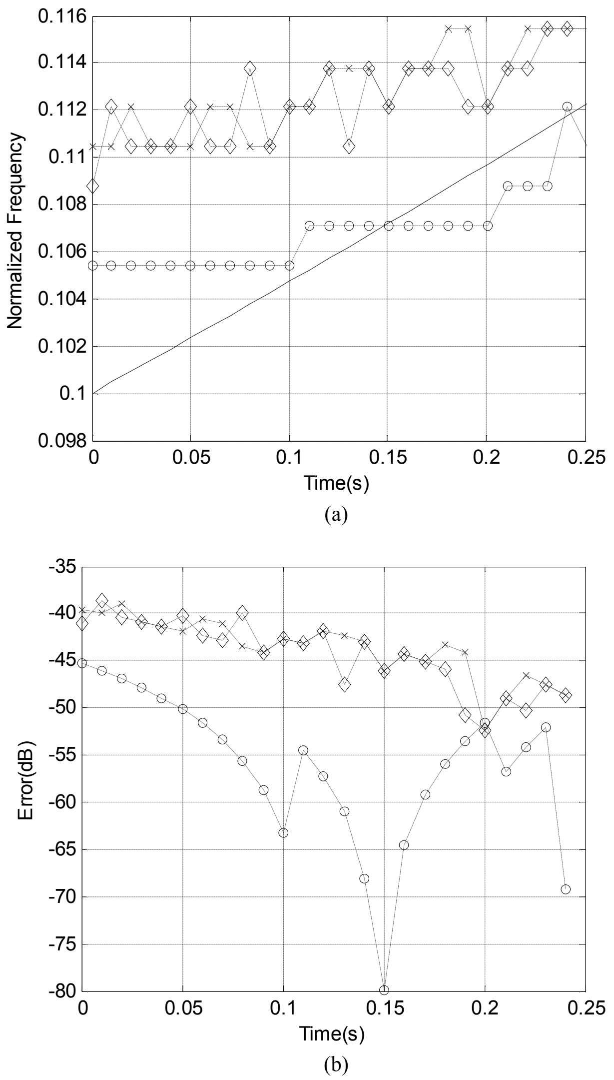


Figure 2. (a) The right border of wavelet ridge for the three traditional methods. (b) Estimation error. Solid line—theoretical result; cross--zero padding; diamond-- periodic extension; circle-- symmetric extension.

III. Border Effects Alleviated via Fourier Extension

As discussed, every traditional method may have drawbacks. We should choose one according to the practical application. In other words, the extension methods should be consistent with the object to be analyzed. The choice of the border extension method depends on the morphology of the end points of the data sequence. When the object of analysis and processing is a signal with time-varying frequency content, we should choose a method which best matches the signals. The extension part should be consistent with the characteristics of the original part in order to alleviate the impact of artificial extension on features of the original signal. We expect the extension will be smooth and represent the past or future of signal that is nonlinear and nonstationary. In additional, wavelet transform should be computationally efficient depends on the particular application. Real-time applications would need to choose border extension methods that provide fast transform implementations.

A. Principle of Fourier Extension Method

Let us now consider the same test signal that has been used in Section II. We can see from Fig. 1(a) that the signal contains many harmonic vibrations. The traditional extension schemes cannot reflect such waving feature. It is nature to think of employing Fourier series technical to extension signal. Because Fourier series can be used to represent a signal in terms of the harmonics it is composed of. Although Fourier series is periodic, as we will see later, we only require a small segment of signal to achieve extension process which can be assumed to be periodic. In this paper, we call this novel scheme the Fourier series extension. The main principle of the Fourier series extension scheme is to fit a Fourier series model to the border points of the signal and then extrapolate that Fourier series at both ends.

Specifically, the procedure of Fourier series extension can be described as following:

1) The Fourier series model is given by

$$y(t) = a_0 + \sum_{i=1}^{m} a_i \cos(\omega_i t) + b_i \cos(\omega_i t) \qquad (2)$$

where a_0 is a constant term in the signal, both a_i , b_i and ω_i are parameters that need to be estimated by the fit, m is the number of harmonics in the data.

Because the above Fourier series model is nonlinear, the first step of Fourier series extension is to perform data transformations to obtain a linear or simple model.

2) Fourier series fitting process involves finding the above model parameters to minimize the summed square of residual defined as the difference between the real date value s and the fitted response value y . This approach is referred as least-squares method.

3) After completing the parameters estimate, the resulting Fourier series is extended to define the data beyond the borders so that the convolution can be calculated.

B. Design of Fourier Extension

When performing the Fourier series extension, several important problems must be considered.

1) The choice of fitting number

The number of the border samples M to be fitted should be chosen carefully. If M is too small, the fitting result could not give a good represent of the characteristics of the original signal. On the other hand, a large M will lead to extra computation. In some applications, e.g., subband coding, M is equivalent to the length of the filters. As we will discuss later, in the application of time-frequency analysis using wavelet with an infinite support, M is determined by the extent of the border effects region.

2) Periodic problem

Fourier series extension does not require the signal to represent a periodic function because we only choose border parts of the signal to perform extension which could be considered as a segment of a periodic function. However, if the data presented are assumed to represent a full cycle of periodic function, then many terms of a Fourier series are needed to achieve fitting. Therefore it is necessary to think in terms of the data representing only a partial segment of one complete periodic cycle so that only a few terms can give a good fit.

C. Properties of Fourier Extension

Unlike the traditional extension methods, the advantage of Fourier series extension is that it avoids the artificial discontinuities at the borders neither in the extension signal nor in its derivatives. So it prevents the appearance of large wavelet transform values at the ends. Therefore, Fourier series extension is a smooth extension. Moreover, using Fourier series to represent signal could well fit the fluctuation features of some very common signals in the areas of diagnosis and monitoring.

It should be noted that there is another smooth extension named polynomial extension. But it needs a higher degree polynomial to fit the fluctuation features compared with the Fourier series extension which has much less computational complexity and provides fast transform implementations.

D. Numerical Examples

Some results that illustrate the performance of our method are shown. Fig. 3 depicts the extension results (only the left border) using Fourier series extension applied to the same signal in Section II. Fourier series extension displays both smooth and a good ‘explanation’ of signal itself. As we have done in the previous section, the error between theoretical and estimated wavelet ridge are shown in Fig. 4(b) to illustrate the effect of Fourier series extension, compared with that obtained by symmetric extension which is superior over the other two traditional methods. It can be clearly observed that less artificial values appear when using our Fourier series extension techniques. In this example, we conclude that our Fourier series extension method provides the best results.

IV. Extent of Border Effects Region

It is important to examine the extent of the border effects region with appearance of the artificial components. The border effects region consists of a segment of transform results where the wavelet coefficients are calculated from the part of signal which contains the extension data. The range of extension would significantly depend on this issue.

A. Mean Square Definition of the Duration of Wavelet

As mentioned in the introduction, border effects root from wavelets analysis window extending beyond the data. Thus, the extent of the border effects region is relative to the width of wavelets analysis window, i.e. the duration of the wavelet.

For the solution of this problem, let take Morlet wavelet for example. Morlet wavelet is a complex sine wave localized with a Gaussian envelope given by

$$\psi(t)=\frac{1}{\sqrt{\pi\gamma_b}}\bullet e^{j2\pi\gamma_c-(t^2/\gamma_b)} \quad (3)$$

where γ_b is a bandwidth parameter defined as the variance

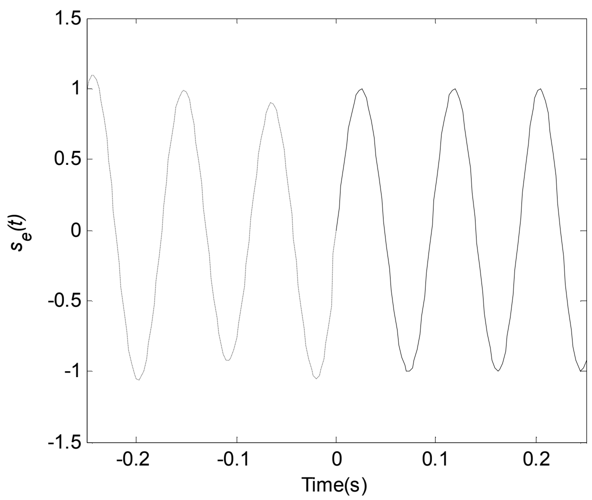
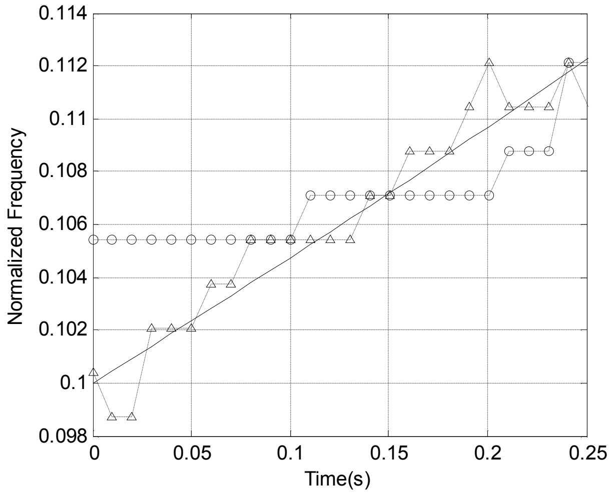
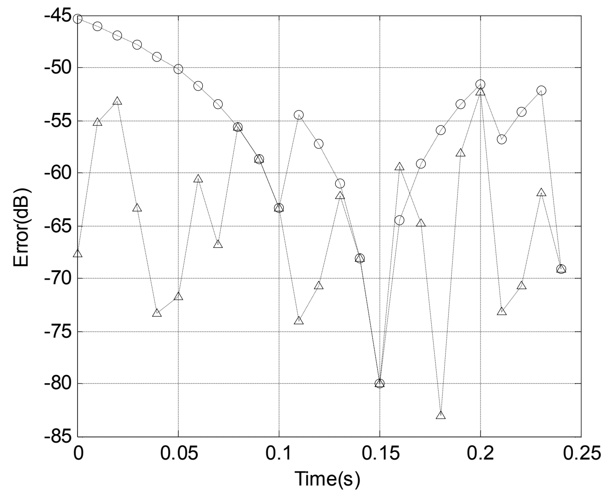


Figure 3. Extension results of the left end of the signal. Solid line—original signal; dotted line—extension segment.



(a)



(b)

Figure 4. (a) Border effects reduction comparison. (b) Error of wavelet ridge estimation. Circle--symmetric extension; triangle—Fourier series extension.

of the Fourier transform of the Morlet wavelet and γ_c denotes the wavelet center frequency.

In fact, the strict duration of the Morlet wavelet is not a compact interval but the entire time axis. Hence, a duration where most of the signal energy is contained might be accepted as a practical measure of the signal duration. A widely used definition of signal duration Δt , proposed by Gabor [25] is

$$\Delta t = \sqrt{\frac{\int_{-\infty}^{\infty} t^2 |s(t)|^2 \, dt}{\int_{-\infty}^{\infty} |s(t)|^2 \, dt}} \tag{4}$$

which is a mean square definition. Based on this definition, the duration for a scaled Morlet wavelet at scale a can be written as

$$\Delta t = \frac{\sqrt{\gamma_b}}{2} a \tag{5}$$

To examine the relationship between the extent of the border effects region and wavelet duration, we consider a simple sine signal with frequency $f = 8Hz$. The transform result of this signal using the wavelet in Fig. 5(a) at scale $a = 1$ is plotted in Fig. 5(b). Due to the impact of border effects, the transform result which should be constant with time represents a curve at both ends. The length of the curve is the actual extent of border effects. It is much longer, not equivalent, than the duration of the wavelet defined in (3).This is due to the fact that the interval $[-\Delta t, \Delta t]$ just contains roughly 53% of the entire signal energy in this case of Morlet wavelet as shown in the Fig. 5(a). Hence, the definition of duration should be modified in order to be applied to the measure of the extent of border effects.

B. Modified Definition of the duration of wavelet

In this paper, we define a new duration Δt_e where a dominating fraction of the signal energy occurs.

Denote the ratio of integral of the modulus of Morlet wavelet over the interval $[-\Delta t_e, \Delta t_e]$ to over the entire time axis as η , then

$$\eta = \frac{\int_{-\Delta t_e}^{\Delta t_e} |\psi(a,t)| \, dt}{\int_{-\infty}^{\infty} |\psi(a,t)| \, dt} \tag{6}$$

The numerator of the (5) is given by

$$\int_{-\Delta t_e}^{\Delta t_e} |\psi(a,t)| \, dt = \int_{-\Delta t_e}^{\Delta t_e} \frac{1}{\sqrt{\pi \gamma_b a}} e^{\frac{-t^2}{\gamma_b a^2}} dt$$
$$= \sqrt{a} erf\left(\frac{\Delta t_e}{a\sqrt{\gamma_b}}\right) \tag{7}$$

The denominator can be written as

$$\int_{-\infty}^{\infty} \psi(a,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi \gamma_b a}} e^{\frac{-t^2}{\gamma_b a^2}} dt = \sqrt{a} \tag{8}$$

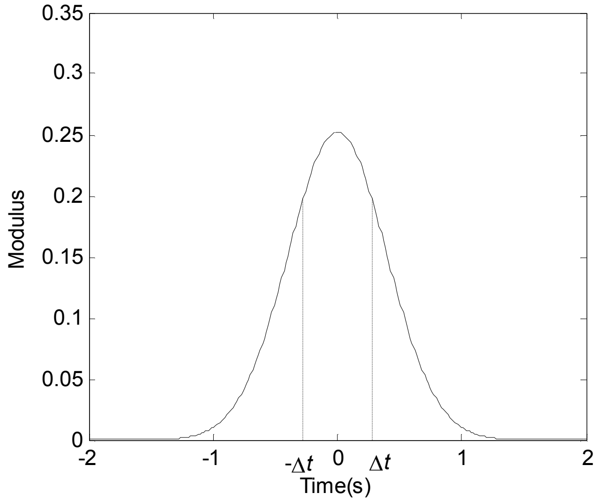
Substituting (6) and (7) into (5) yields

$$\eta = erf\left(\frac{\Delta t_e}{a\sqrt{\gamma_b}}\right) \tag{9}$$

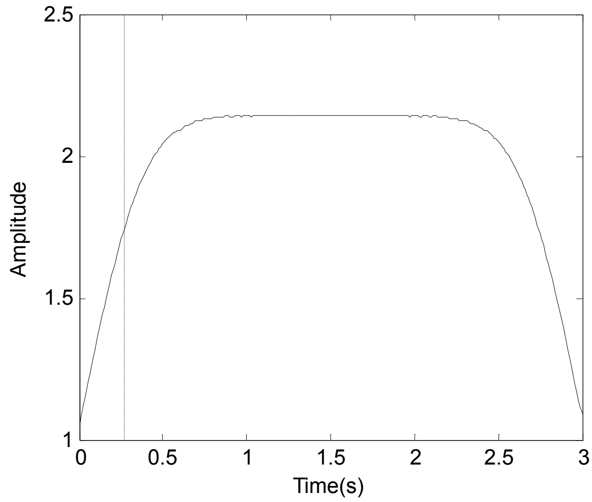
where $erf(x)$ is the error function defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{10}$$

Then, the modified duration is defined as Δt_e that satisfies the formula in (8).

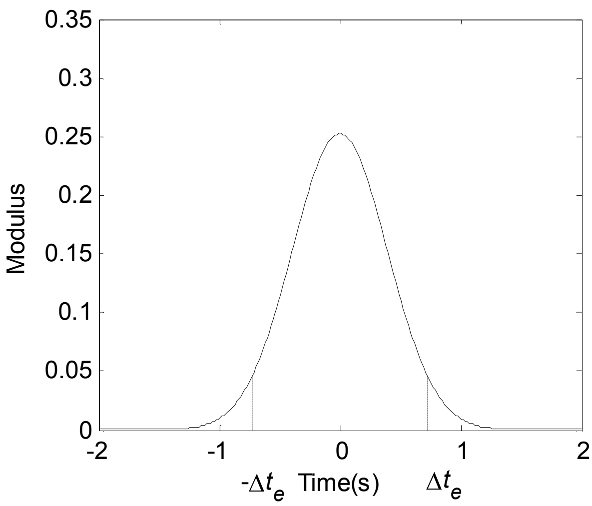


(a)



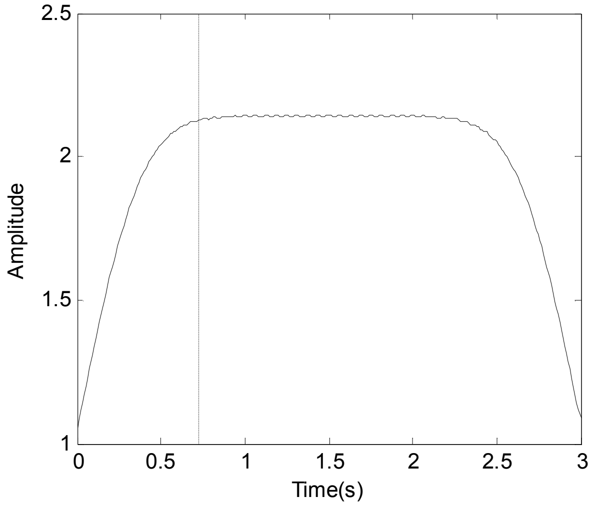
(b)

Figure 5. (a)Duration defined by (3) in the modulus of a Morlet wavelet and (b) related extent of border effects region in the wavelet transform of a sine signal with $f = 8Hz$. The left side of dotted line represents the extent of border effects region.



(a)

If we set the parameter $\eta = 0.9$, which means the new duration containing majority effective part of the wavelet, the duration of the same Morlet wavelet becomes $[-0.73, 0.73]$ (s) which approaches to the extent of border effects region. Since such a definition yields an interval



(b)

Figure 6. (a)Duration defined by (8) in the modulus of a Morlet wavelet and (b) related extent of border effects region in the wavelet transform of a sine signal with $f = 8Hz$. The left side of dotted line represents the border effects region.

containing dominating fraction of the wavelet energy. Fig. 6 shows the new duration. In Fig. 6(a), we can observe that the new duration of wavelet is much more reasonable since the extent of border effects region obtained from it covers the whole distortion part in the wavelet transform.

Note that according to (8), the extent of border effects region increases linearly with scale parameter a . Furthermore, the degree of the border effects will become less as closing to the interior of the signal where the calculation of the wavelet coefficients involves less artificial data.

V. Conclusion

Border effects often arise when a convolution is performed on finite-length signals. We have discussed the problem of dealing with the border effects in the application of nonlinear time-varying analysis. A smooth extension scheme using Fourier series to avoid distortion appearing at the borders was proposed. Numerical examples results depicted that Fourier series extension produces lower error in comparison to traditional methods including zero padding, periodic extension and symmetric extension for the chosen FM signal. A strict definition of duration of wavelet was introduced to measure the extent of the border effects region which would be applied to the procedure of Fourier extension. An example based on Morlet wavelet was used to validate the theoretical derivations.

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